

Multi-Time Dependent Sprays and Harmonic Maps on $J^1(T, M)$

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Abstract

It is known that the jet fibre bundle of order one $J^1(T, M)$ is a basic object in the study of classical and quantum field theories. In order to develop a subsequent multi-time dependent Lagrangian theory of physical fields on $J^1(T, M)$, we need to generalize the main geometrical objects used in the classical rheonomic Lagrangian theory. In this direction, Section 1 presents the main properties of the differentiable structure of the jet fibre bundle of order one. Section 2 studies an important collection of geometrical objects on $J^1(T, M)$ as d-tensors, temporal and spatial sprays and the harmonic maps induced by these sprays, which naturally generalize analogous objects on $R \times TM$, the natural house of the time-dependent Lagrangian field theory [6]. Section 3 studies the nonlinear connections Γ on $J^1(T, M)$, and discuss their relation with the temporal and spatial sprays. Section 4 opens the problem of prolongation of vector fields from $T \times M$ to 1-jet space $J^1(T, M)$, using adapted bases.

Mathematics Subject Classification (1991): 53C07, 53C43, 53C99

Key words: 1-jet fibre bundle, d-tensors, temporal and spatial sprays, harmonic maps, nonlinear connection.

1 The jet fibre bundle $J^1(T, M)$

Let us consider the smooth manifolds T and M of dimension p , respectively n , coordinated by $(t^\alpha)_{\alpha=\overline{1,p}}$, respectively $(x^i)_{i=\overline{1,n}}$. We remark that, throughout this paper, the set $\{1, 2, \dots, p\}$ is indexed by $\alpha, \beta, \gamma, \dots$, and the set $\{1, 2, \dots, n\}$ is indexed by i, j, k, \dots .

Now, let (t_0, x_0) be an arbitrary point of the product manifold $T \times M$. We denote $C^\infty(T, M)$ the set of all smooth maps between T and M and define the equivalence relation

$$(1.1) \quad f \sim_{(t_0, x_0)} g \Leftrightarrow \begin{cases} f(t_0) = g(t_0) = x_0 \\ df_{t_0} = dg_{t_0}, \end{cases}$$

on $C^\infty(T, M)$. For every $f, g \in C^\infty(T, M)$, the relation $f \sim_{(t_0, x_0)} g$ can be expressed locally by

$$(1.2) \quad \begin{cases} x^i(t_0^\beta) = y^i(t_0^\beta) = x_0^i \\ \frac{\partial x^i}{\partial t^\alpha}(t_0^\beta) = \frac{\partial y^i}{\partial t^\alpha}(t_0^\beta), \end{cases}$$

where $t^\beta(t_0) = t_0^\beta$, $x^i(x_0) = x_0^i$, $x^i = x^i \circ f$ and $y^i = x^i \circ g$. The equivalence class of a smooth map $f \in C^\infty(T, M)$ is denoted by $[f]_{(t_0, x_0)} = \{g \in C^\infty(T, M) \mid g \sim_{(t_0, x_0)} f\}$. If the quotient $J_{t_0, x_0}^1(T, M) = C^\infty(T, M) / \sim_{(t_0, x_0)}$ is the factorization by the equivalence relation " $\sim_{(t_0, x_0)}$ ", we build the total space of the 1-jet set, taking

$$(1.3) \quad J^1(T, M) = \bigcup_{(t_0, x_0) \in T \times M} J_{t_0, x_0}^1(T, M).$$

Let us organize the total space of 1-jets $J^1(T, M)$ as a vector bundle over the base space $T \times M$. To do this fact, we start with a smooth map $f \in C^\infty(T, M)$, $(t^1, \dots, t^p) \rightarrow (x^1(t^1, \dots, t^p), \dots, x^n(t^1, \dots, t^p))$, and expand the maps x^i using Taylor formula around the point $(t_0^1, \dots, t_0^p) \in R^p$. We obtain

$$x^i(t^1, \dots, t^p) = x_0^i + (t^\alpha - t_0^\alpha) \frac{\partial x^i}{\partial t^\alpha}(t_0^1, \dots, t_0^p) + \mathcal{O}(2), \quad \|(t^1 - t_0^1, \dots, t^p - t_0^p)\| < \varepsilon.$$

Considering the smooth map $\tilde{f} \in C^\infty(T, M)$ defined by the local functions set

$$\tilde{x}^i(t^1, \dots, t^p) = x_0^i + (t^\alpha - t_0^\alpha) \frac{\partial x^i}{\partial t^\alpha}(t_0^1, \dots, t_0^p), \quad \|(t^1 - t_0^1, \dots, t^p - t_0^p)\| < \varepsilon,$$

we deduce that $\tilde{f} \sim_{(t_0, x_0)} f$, that is, the linear affine approximation \tilde{f} of f is a convenient representative of the equivalence class $[f]_{(t_0, x_0)}$.

Let $\pi : J^1(T, M) \rightarrow T \times M$ be the projection defined by $\pi([f]_{(t_0, x_0)}) = (t_0, f(t_0))$. It is obvious that the map π is well defined and surjective. Using this projection, for every local chart $U \times V \subset T \times M$ on the product manifold $T \times M$, we can define the bijection

$$\Phi_{U \times V} : \pi^{-1}(U \times V) \rightarrow U \times V \times R^{np},$$

setting $\Phi_{U \times V}([f]_{(t_0, x_0)}) = (t_0, x_0, \frac{\partial x^i}{\partial t^\alpha}(t_0^\beta))$, $x_0 = f(t_0)$.

In conclusion, the 1-jet set $J^1(T, M)$ can be endowed with a differentiable structure of dimension $p + n + pn$, such that the maps $\Phi_{U \times V}$ to be diffeomorphisms. We emphasize that the local coordinates on $J^1(T, M)$ are $(t^\alpha, x^i, x_\alpha^i)$, where

$$(1.4) \quad \begin{cases} t^\alpha([f]_{(t_0, x_0)}) = t^\alpha(t_0) \\ x^i([f]_{(t_0, x_0)}) = x^i(x_0) \\ x_\alpha^i([f]_{(t_0, x_0)}) = \frac{\partial x^i}{\partial t^\alpha}(t_0^\beta). \end{cases}$$

In the above coordinates on $J^1(T, M)$, the projection $\pi : J^1(T \times M) \rightarrow T \times M$ has the local expression $\pi(t^\alpha, x^i, x_\alpha^i) = (t^\alpha, x^i)$. Moreover, the differential π_* of the map π is locally determined by the Jacobi matrix

$$\begin{pmatrix} \delta_{\alpha\beta} & 0 & 0 \\ 0 & \delta_{ij} & 0 \end{pmatrix} \in M_{p+n, p+n+pn}.$$

It follows that π_* is a surjection (rank $\pi_* = p + n$), and therefore the projection π is a submersion. Consequently, the 1-jet total space $J^1(T, M)$ becomes a vector bundle over the base space $T \times M$, having the fibre type R^{pn} .

Using 1.4, by a simple direct calculation, we obtain

Proposition 1.1 *The local coordinate transformations $(t^\alpha, x^i, x_\alpha^i) \longleftrightarrow (\tilde{t}^\alpha, \tilde{x}^i, \tilde{x}_\alpha^i)$ of the 1-jet vector bundle $E = J^1(T, M)$ are given by*

$$(1.5) \quad \begin{cases} \tilde{t}^\alpha = \tilde{t}^\alpha(t^\beta) \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{x}_\alpha^i = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial t^\beta}{\partial \tilde{t}^\alpha} x_\beta^j, \end{cases}$$

where $\det(\partial \tilde{t}^\alpha / \partial t^\beta) \neq 0$ and $\det(\partial \tilde{x}^i / \partial x^j) \neq 0$. Consequently, E is always an orientable manifold.

Let us consider the canonical basis $\left\{ \frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x_\alpha^i} \right\}$ of vector fields on E and $\{dt^\alpha, dx^i, dx_\alpha^i\}$ its dual basis of 1-forms.

Proposition 1.2 *Changing the coordinates on E , the following transformation rules are true:*

$$(1.6) \quad \begin{cases} \frac{\partial}{\partial t^\alpha} = \frac{\partial \tilde{t}^\beta}{\partial t^\alpha} \frac{\partial}{\partial \tilde{t}^\beta} + \frac{\partial \tilde{x}_\beta^j}{\partial t^\alpha} \frac{\partial}{\partial \tilde{x}_\beta^j} \\ \frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{x}_\beta^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}_\beta^j} \\ \frac{\partial}{\partial x_\alpha^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial t^\alpha}{\partial \tilde{t}^\beta} \frac{\partial}{\partial \tilde{x}_\beta^j}, \end{cases}$$

$$(1.7) \quad \begin{cases} dt^\alpha = \frac{\partial t^\alpha}{\partial \tilde{t}^\beta} d\tilde{t}^\beta \\ dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j \\ dx_\alpha^i = \frac{\partial x_\alpha^i}{\partial \tilde{t}^\beta} d\tilde{t}^\beta + \frac{\partial x_\alpha^i}{\partial \tilde{x}^j} d\tilde{x}^j + \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{t}^\beta}{\partial t^\alpha} d\tilde{x}_\beta^j. \end{cases}$$

Some physical aspects.

At the end of this Section, we should like to expose certain physical aspects of the jet vector bundle of order one that we consider very eloquent for the subsequent theory.

Thus, from physical point of view, we regard the space T as a "temporal" manifold or a "multi-time" while the manifold M is regarded as a "spatial" one. The vector bundle $J^1(T, M) \rightarrow T \times M$ is regarded as a *bundle of configurations*, in mechanics terms, and its elements $[f]$ are regarded as classes of "parametrized sheets".

In order to motivate the terminology used, we study more deeply the jet vector bundle of order one, in the particular case $T = R$ (i. e., the usual time axis represented by the set of real numbers). Let us suppose that $J^1(R, M) \equiv R \times TM$ is coordinated by (t, x^i, y^i) . The gauge group of the bundle

$$(1.8) \quad \pi : J^1(R, M) \rightarrow R \times M, \quad (t, x^i, y^i) \rightarrow (t, x^i),$$

is given by

$$(1.9) \quad \begin{cases} \tilde{t} = \tilde{t}(t) \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{d\tilde{t}} y^j. \end{cases}$$

We remark that the form of this gauge group stands out by the *relativistic* character of the time t . For that reason, we consider that the jet fibre bundle of order one $J^1(R, M)$ is the natural bundle of configurations of the *relativistic rheonomic Lagrangian mechanics* [11].

Comparatively, in the *classical rheonomic Lagrangian mechanics* [6], the bundle of configurations is the fibre bundle

$$(1.10) \quad \pi : R \times TM \rightarrow M, (t, x^i, y^i) \rightarrow (x^i),$$

whose geometrical invariance group is

$$(1.11) \quad \begin{cases} \tilde{t} = t \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j. \end{cases}$$

Obviously, the structure of the gauge group 1.11 emphasizes the *absolute* character of the time t from the classical rheonomic Lagrangian mechanics. At the same time, we point out that the gauge group 1.11 is a subgroup of 1.9. In other words, the gauge group of the jet bundle of order one, from the relativistic rheonomic Lagrangian mechanics, is more general than that used in the classical rheonomic Lagrangian mechanics, which ignores the temporal reparametrizations.

Finally, we invite the reader to compare both the classical and relativistic rheonomic Lagrangian mechanics developed in [6] and [11].

2 d-Tensors. Multi-time dependent sprays. Harmonic maps

It is well known the importance of the tensors in the development of a geometry on a fibre bundle. In the study of the 1-jet fibre bundle, a central role is played by the *distinguished tensors* or *d-tensors*.

Definition 2.1 A geometrical object $D = (D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\nu)\dots})$, on the 1-jet vector bundle E , whose local components verify the following rules of transformation,

$$(2.1) \quad D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\nu)\dots} = \tilde{D}_{\varepsilon r(\mu)(s)\dots}^{\delta p(m)(\eta)\dots} \frac{\partial t^\alpha}{\partial \tilde{t}^\delta} \frac{\partial x^i}{\partial \tilde{x}^p} \frac{\partial x^j}{\partial \tilde{x}^m} \frac{\partial \tilde{t}^\mu}{\partial t^\beta} \frac{\partial \tilde{t}^\varepsilon}{\partial t^\gamma} \frac{\partial \tilde{x}^r}{\partial x^k} \frac{\partial \tilde{x}^s}{\partial x^l} \frac{\partial t^\nu}{\partial \tilde{t}^\eta} \dots,$$

is called a *d-tensor field*.

Remarks 2.1 i) The utilization of parentheses for certain indices of the local components $D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\nu)\dots}$ will be motivated at the end of the Section 3 of this paper, before the introduction of a nonlinear connection Γ on E together with its adapted bases of vector and covector fields (see Remark 3.2).

ii) A d-tensor field D on $E = J^1(T, M)$ can be viewed like an object defined on $T \times M$ which depends on *partial derivatives* or *partial directions* x_α^i .

Examples 2.1 i) If $L : E \rightarrow R$ is a multi-time Lagrangian function with partial derivatives of order one, the local components

$$(2.2) \quad G_{(i)(j)}^{(\alpha)(\beta)} = \frac{1}{2} \frac{\partial^2 L}{\partial x_\alpha^i \partial x_\beta^j}$$

represent a d-tensor field on E . We point out that taking $T = R$ and L a regular time-dependent Lagrangian, the d-tensor field $G_{(i)(j)}^{(1)(1)}(t, x^i, y^i)$ is a natural generalization of that so-called metrical d-tensor field $g_{ij}(t, x, y)$ of a classical rheonomic Lagrange space $RL^n = (M, L(t, x^i, y^i))$ [6].

ii) The geometrical object $\mathbf{C} = (\mathbf{C}_{(\alpha)}^{(i)})$, where $\mathbf{C}_{(\alpha)}^{(i)} = x_\alpha^i$, represent a d-tensor field on E . This is called the *canonical Liouville d-tensor* on the 1-jet vector bundle E . We emphasize that this d-tensor field naturally generalizes the Liouville d-vector field $\mathbf{C} = y^i \frac{\partial}{\partial y^i}$ used in [6].

iii) Let $h_{\alpha\beta}$ be a semi-Riemannian metric on the temporal manifold T . The geometrical object $L = (L_{(\alpha)\beta\gamma}^{(i)})$, where $L_{(\alpha)\beta\gamma}^{(i)} = h_{\beta\gamma} x_\alpha^i$, is a d-tensor field which is called the *Liouville d-tensor associated to the metric h* .

iv) Using the preceding metric h , we construct the d-tensor $J = (J_{(\alpha)\beta j}^{(i)})$, where $J_{(\alpha)\beta j}^{(i)} = h_{\alpha\beta} \delta_j^i$. This d-tensor is called the *h -normalization d-tensor* of the jet bundle E . Note that the h -normalization d-tensor of $J^1(T, M)$ is a natural generalization of the tangent structure J from the Lagrange geometry [6].

It is obvious that any d-tensor on E is a tensor on E . Conversely, this is not true. As examples, we will build two tensors which are not d-tensors. We refer to notions of temporal and spatial sprays which allow the generalization of the notion of time-dependent spray used in [6], [14].

Definition 2.2 A global tensor H , expressed locally by

$$(2.3) \quad H = \delta_\alpha^\beta dt^\alpha \otimes \frac{\partial}{\partial t^\beta} - 2H_{(\beta)\alpha}^{(j)} dt^\alpha \otimes \frac{\partial}{\partial x_\beta^j},$$

is called a *temporal spray* on E .

Taking into account that a temporal spray is a global tensor on E , by a direct calculation, we deduce

Proposition 2.1 i) The components $H_{(\beta)\alpha}^{(j)}$ of the temporal spray H transform by the rules

$$(2.4) \quad 2\tilde{H}_{(\mu)\gamma}^{(k)} = 2H_{(\beta)\alpha}^{(j)} \frac{\partial t^\alpha}{\partial \tilde{t}^\gamma} \frac{\partial \tilde{x}^k}{\partial x_\beta^j} \frac{\partial t^\beta}{\partial \tilde{t}^\mu} - \frac{\partial t^\alpha}{\partial \tilde{t}^\gamma} \frac{\partial \tilde{x}_\mu^k}{\partial t^\alpha}.$$

ii) Conversely, to give a temporal spray on E is equivalent to give a set of local functions $H = (H_{(\beta)\alpha}^{(j)})$ which transform by 2.4.

iii) The global tensor

$$H = H_\alpha^\beta dt^\alpha \otimes \frac{\partial}{\partial t^\beta} - 2H_{(\beta)\alpha}^{(j)} dt^\alpha \otimes \frac{\partial}{\partial x_\beta^j}$$

is a temporal spray iff $J_{(\beta)\alpha i}^{(j)} H_\gamma^\alpha = J_{(\beta)\gamma i}^{(j)}$, where J is the normalization d-tensor of the fibre bundle E associated to an arbitrary semi-Riemannian metric h on T .

The previous proposition allows us to offer the following important example of temporal spray. The importance of this temporal spray is determined by its using in the description of the classical harmonic maps between two semi-Riemannian manifolds [3].

Example 2.2 Using the transformation rules of the Christoffel symbols $H_{\beta\gamma}^\alpha$ attached to a semi-Riemannian metric $h_{\alpha\beta}$ on T , we deduce that the components $2H_{(\beta)\alpha}^{(j)} = -H_{\alpha\beta}^\gamma x_\gamma^j$ represent a temporal spray on E . This is called the *canonical temporal spray associated to the metric h* .

Definition 2.3 A global tensor G , locally defined by

$$(2.5) \quad G = x_\alpha^i dt^\alpha \otimes \frac{\partial}{\partial x^i} - 2G_{(\beta)\alpha}^{(j)} dt^\alpha \otimes \frac{\partial}{\partial x_\beta^j},$$

is called a *spatial spray* on E .

As in the case of the temporal spray, we can prove without difficulties the following statements.

Proposition 2.2 The components $G_{(\beta)\alpha}^{(j)}$ of the spatial spray G transform by the rules

$$(2.6) \quad 2\tilde{G}_{(\mu)\gamma}^{(k)} = 2G_{(\beta)\alpha}^{(j)} \frac{\partial t^\alpha}{\partial \tilde{t}^\gamma} \frac{\partial \tilde{x}^k}{\partial x^j} \frac{\partial t^\beta}{\partial \tilde{t}^\mu} - \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{x}_\mu^k}{\partial x^i} \tilde{x}_\gamma^j.$$

ii) To give a spatial spray is equivalent to give a set of local functions $G = (G_{(\beta)\alpha}^{(j)})$ which change by the law 2.6.

iii) A global tensor on E , defined locally by

$$G = G_\alpha^i dt^\alpha \otimes \frac{\partial}{\partial x^i} - 2G_{(\beta)\alpha}^{(j)} dt^\alpha \otimes \frac{\partial}{\partial x_\beta^j},$$

is a spatial spray iff $J_{(\beta)\alpha i}^{(j)} G_\gamma^i = L_{(\beta)\alpha\gamma}^{(j)}$, where J (resp. L) is the normalization (resp. Liouville) d-tensor associated to an arbitrary semi-Riemannian metric h .

Example 2.3 If γ_{jk}^i are the Christoffel symbols of a semi-Riemannian metric φ_{ij} on the spatial manifold M , the local coefficients $2G_{(\beta)\alpha}^{(j)} = \gamma_{kl}^j x_\alpha^k x_\beta^l$ define a spatial spray which is called the *canonical spatial spray associated to the metric φ* . We point out that this spatial spray is also used in the description of the classical harmonic maps between two semi-Riemannian manifolds [3].

Definition 2.4 A pair (H, G) , which consists of a temporal spray and a spatial one, is called a *multi-time dependent spray* on E .

To characterize the multi-time dependent sprays on E and to underline again the importance of the canonical temporal and spatial sprays attached to the metrics h and φ , we prove the following

Theorem 2.3 Let (T, h) , (M, φ) be semi-Riemannian manifolds and let $H = (H_{(\alpha)\beta}^{(i)})$ (resp. $G = (G_{(\alpha)\beta}^{(i)})$) be an arbitrary temporal (resp. spatial) spray on E . In these conditions, we have

$$(2.7) \quad \begin{cases} H_{(\alpha)\beta}^{(i)} = -\frac{1}{2}H_{\alpha\beta}^{\gamma}x_{\gamma}^i + D_{(\alpha)\beta}^{(i)} \\ G_{(\alpha)\beta}^{(i)} = \frac{1}{2}\gamma_{jk}^i x_{\alpha}^j x_{\beta}^k + F_{(\alpha)\beta}^{(i)}, \end{cases}$$

where $D_{(\alpha)\beta}^{(i)}$, $F_{(\alpha)\beta}^{(i)}$ are certain d-tensors on E .

Proof. The theorem comes from the following true statements:

- i) An affine combination of temporal (spatial) sprays is a temporal (spatial) spray,
- ii) The product between a scalar and a temporal (spatial) spray is a temporal (spatial) spray,
- iii) The difference between two temporal (spatial) sprays is a d-tensor. ■

In order to generalize the notion of path of a spray from Lagrangian geometry, we fixe $h_{\alpha\beta}$ a semi-Riemannian metric on the temporal manifold T . In this context, we give the following

Definition 2.5 A geometrical object $H = (H^k)$ (resp. $G = (G^k)$) is called a *temporal* (resp. *spatial*) *h-spray* if the local components modify by the rules

$$(2.8) \quad 2\tilde{H}^k = 2H^j \frac{\partial \tilde{x}^k}{\partial x^j} - \tilde{h}^{\gamma\mu} \frac{\partial t^{\alpha}}{\partial \tilde{t}^{\gamma}} \frac{\partial \tilde{x}_{\mu}^k}{\partial t^{\alpha}},$$

repectively

$$(2.9) \quad 2\tilde{G}^k = 2G^j \frac{\partial \tilde{x}^k}{\partial x^j} - \tilde{h}^{\gamma\mu} \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{x}_{\mu}^k}{\partial x^i} \tilde{x}_{\gamma}^j.$$

Example 2.4 Starting with $H = (H_{(\alpha)\beta}^{(i)})$ (resp. $G = (G_{(\alpha)\beta}^{(i)})$) like a temporal (resp. spatial) spray, the entity $H = (H^i)$ (resp. $G = (G^i)$), where $H^i = h^{\alpha\beta} H_{(\alpha)\beta}^{(i)}$ (resp. $G^i = h^{\alpha\beta} G_{(\alpha)\beta}^{(i)}$), represents a temporal (resp. spatial) *h-spray* which will be called the *h-trace of the temporal* (resp. *spatial*) spray H (resp. G). Particularly, the components $H^k = -h^{\alpha\beta} H_{\alpha\beta}^{\gamma} x_{\gamma}^k$ (resp. $G^k = h^{\alpha\beta} \gamma_{ij}^k x_{\alpha}^i x_{\beta}^j$) represent the *canonical temporal* (resp. *spatial*) *h-spray attached to the metric h* (resp. φ).

The previous example show that the *h-trace* of a temporal or a spatial spray represents a temporal or a spatial *h-spray*. Conversely, we prove the following

Theorem 2.4 If $\dim T = 1$, any temporal (spatial) *h-spray* is the *h-trace* of a unique temporal (spatial) spray.

Proof. Let $G = (G^k)$ be a spatial *h-spray*. We denote $G_{(1)1}^{(k)} = h_{11} G^k$. Obviously, the relation $G^k = h^{11} G_{(1)1}^{(k)}$ is true. In these conditions, using the transformation rules 2.9, we deduce

$$2\tilde{G}_{(1)1}^{(k)} = 2G_{(1)1}^{(j)} \frac{\partial \tilde{x}^k}{\partial x^j} \left(\frac{d\tilde{t}}{dt} \right)^2 - \frac{dt}{d\tilde{t}} \frac{dy^k}{dt}.$$

This means that $G = (G_{(1)1}^{(k)})$ is a spatial spray. The uniqueness is clear.

By analogy, we treat the case of the temporal h -sprays, taking $H^k = h^{11}H_{(1)1}^{(k)}$, where $H_{(1)1}^{(k)} = h_{11}H^k$. ■

Remark 2.2 The previous theorem shows that, in the case $\dim T = 1$, there is a 1-1 correspondence between sprays and h -sprays while, for $\dim T \geq 2$, this statement is not true.

In the sequel, let us fix a temporal spray $H = (H_{(\alpha)\beta}^{(i)})$ and a spatial spray $G = (G_{(\alpha)\beta}^{(i)})$ on E . The following notions show that the 1-jet fibre bundle is the natural house for important objects with geometrical and physical meaning.

Definition 2.6 A solution $f \in C^\infty(T, M)$ of the PDEs system of order two

$$(2.10) \quad x_{\alpha\beta}^i + G_{(\alpha)\beta}^{(i)} + G_{(\beta)\alpha}^{(i)} + H_{(\alpha)\beta}^{(i)} + H_{(\beta)\alpha}^{(i)} = 0,$$

where the map f is locally expressed by $(t^\alpha) \rightarrow (x^i(t^\alpha))$ and $x_{\alpha\beta}^i = \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta}$, is called an *affine map of the multi-time dependent spray* (H, G) .

Remark 2.3 A reason which offers the naturalness of the notion of an affine map of a multi-time dependent spray on E , is that, in the particular case $T = R$, the equations of the affine maps generalize the equations of the paths of a time-dependent spray from the rheonomic Lagrangian geometry [2].

Example 2.5 Considering the canonical multi-time dependent spray

$$(2.11) \quad \begin{cases} H_{(\alpha)\beta}^{(i)} = -\frac{1}{2}H_{\alpha\beta}^\gamma x_\gamma^i \\ G_{(\alpha)\beta}^{(i)} = \frac{1}{2}\gamma_{jk}^i x_\alpha^j x_\beta^k, \end{cases}$$

the equations of the affine maps of this spray reduce to

$$(2.12) \quad x_{\alpha\beta}^i - H_{\alpha\beta}^\gamma x_\gamma^i + \gamma_{jk}^i x_\alpha^j x_\beta^k = 0,$$

that is, the equations whose solutions are exactly the maps $f \in C^\infty(T, M)$ which carry the geodesics of $(T, h_{\alpha\beta})$ into the geodesics of the space (M, φ_{ij}) .

Taking $h = (h_{\alpha\beta}(t))$ a temporal semi-Riemannian metric and doing a contraction by $h^{\alpha\beta}$ in 2.10, we can introduce the next

Definition 2.7 A map $f \in C^\infty(T, M)$ is called a *harmonic map of the multi-time dependent spray* (H, G) , with respect to the semi-Riemannian metric h , if f is a solution of the PDEs system of order two

$$(2.13) \quad h^{\alpha\beta} \{x_{\alpha\beta}^i + 2G_{(\alpha)\beta}^{(i)} + 2H_{(\alpha)\beta}^{(i)}\} = 0.$$

Example 2.6 Particularly, in the case of the canonical multi-time dependent spray of preceding example, we recover the classical notion of harmonic map between the

semi-Riemannian manifolds (T, h) and (M, φ) [3]. This fact points out that our generalization of the classical notion of harmonic map is a natural one.

Remarks 2.4 i) It is obvious that the affine map of a multi-time dependent spray (H, G) is a harmonic map of the same spray with respect to any semi-Riemannian metric $h_{\alpha\beta}$ on the temporal space T .

ii) In the particular case $(T, h) = (R, \delta)$, the notions of harmonic map and affine map identify. Consequently, both notions naturally generalize that so-called a path of a time-dependent spray, used in [6].

Let us denote $S_{(\alpha)\beta}^{(i)} = G_{(\alpha)\beta}^{(i)} + H_{(\alpha)\beta}^{(i)} + \frac{1}{2}H_{\alpha\beta}^{\gamma}x_{\gamma}^i$ and $S^i = h^{\alpha\beta}S_{(\alpha)\beta}^{(i)}$. Using the theorem 2.3, we deduce that $S = (S^i)$ is a spatial h -spray. In this context, we obtain the without difficulties the following

Theorem 2.5 *The equations of the harmonic maps of the multi-time dependent spray (G, H) , with respect to the semi-Riemannian metric h , can be rewritten in the Poisson form*

$$(2.14) \quad \Delta_h x^i + 2S^i = 0,$$

where $\Delta_h x^i = h^{\alpha\beta}(x_{\alpha\beta}^i - H_{\alpha\beta}^{\gamma}x_{\gamma}^i)$.

Remarks 2.5 i) This theorem will play a central role in the development of the subsequent (generalized) metrical multi-time Lagrange theory of physical fields. In this sense, we will prove over there that the Euler-Lagrange equations of a multi-time dependent Lagrangian $\mathcal{L} = L\sqrt{|h|}$, where $L : J^1(T, M) \rightarrow R$ is a *Kronecker h -regular* Lagrange function, can be written in the Poisson form 2.14. Hence, the extremals of \mathcal{L} can be regarded as *harmonic maps*, in our sense, offering them a profound geometrical and physical character. For more details, see [8], [10].

ii) On the other hand, the same theorem will be used to offer a beautiful geometrical interpretation of solutions of PDEs, in metrical multi-time Lagrangian geometry terms [18]. In this fashion, we will offer a final answer to the Udriște-Neagu open problem [9], [17], whose essential physical aspects are presented in [15], [16].

3 Nonlinear connections

The form of the coordinate changes on $E = J^1(T, M)$ determines complicated rules of transformation of the local components of diverse geometrical objects of this space. This motivates the introduction of *nonlinear connection* which induces *adapted bases*. These bases have the quality to simplify the transformation rules of the components of the geometrical objects taken in study.

With a view to doing this, we take $u \in E$ and consider the differential map $\pi_{*,u} : T_u E \rightarrow T_{(t,x)}(T \times M)$ of the canonical projection $\pi : E \rightarrow T \times M$, $\pi(u) = (t, x)$. At the same time, let us consider the vector subspace $V_u = \text{Ker } \pi_{*,u} \subset T_u E$. Because the map $\pi_{*,u}$ is a surjection, we have $\dim_R V_u = pn$, $\forall u \in E$. Moreover, a basis in V_u is determined by $\{\frac{\partial}{\partial x_{\alpha}^i}\}$. In conclusion, the map

$$(3.1) \quad \mathcal{V} : u \in E \rightarrow V_u \subset T_u E$$

is a differential distribution which is called *the vertical distribution* of the 1-jet fibre bundle E .

Definition 3.1 A *nonlinear connection* on E is a differential distribution

$$(3.2) \quad \mathcal{H} : u \in E \rightarrow H_u \subset T_u E$$

which verifies the relation

$$(3.3) \quad T_u E = H_u \oplus V_u, \quad \forall u \in E.$$

The distribution \mathcal{H} is called the *horizontal distribution* on E .

Remarks 3.1 i) The above definition implies that $\dim_R H_u = p + n, \forall u \in E$.

ii) The vector fields set $\mathcal{X}(E)$ can be decompose in the following direct sum $\mathcal{X}(E) = \Gamma(\mathcal{H}) \oplus \Gamma(\mathcal{V})$, where $\Gamma(\mathcal{H})$ (resp. $\Gamma(\mathcal{V})$) is the set of the sections on \mathcal{H} (resp. \mathcal{V}).

Now, supposing that there is a nonlinear connection \mathcal{H} on E , we have the isomorphism

$$(3.4) \quad \pi_{*,u}|_{H_u} : H_u \rightarrow T_{\pi(u)}(T \times M),$$

which allows us to prove the following

Theorem 3.1 i) *There exist the unique horizontal vector fields $\frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i} \in \Gamma(\mathcal{H})$, linearly independent, having the property*

$$(3.5) \quad \pi_* \left(\frac{\delta}{\delta t^\alpha} \right) = \frac{\partial}{\partial t^\alpha}, \quad \pi_* \left(\frac{\delta}{\delta x^i} \right) = \frac{\partial}{\partial x^i}.$$

ii) *The vector fields $\frac{\delta}{\delta t^\alpha}$ and $\frac{\delta}{\delta x^i}$ can be uniquely written in the form*

$$(3.6) \quad \begin{cases} \frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} - M_{(\beta)\alpha}^{(j)} \frac{\partial}{\partial x_\beta^j} \\ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(\beta)i}^{(j)} \frac{\partial}{\partial x_\beta^j}. \end{cases}$$

iii) *The coefficients $M_{(\beta)\alpha}^{(j)}$ and $N_{(\beta)i}^{(j)}$ modify by the rules*

$$(3.7) \quad \begin{cases} \tilde{M}_{(\beta)\mu}^{(j)} \frac{\partial \tilde{t}^\mu}{\partial t^\alpha} = M_{(\gamma)\alpha}^{(k)} \frac{\partial \tilde{x}^j}{\partial x^k} \frac{\partial t^\gamma}{\partial \tilde{t}^\beta} - \frac{\partial \tilde{x}_\beta^j}{\partial t^\alpha} \\ \tilde{N}_{(\beta)k}^{(j)} \frac{\partial \tilde{x}^k}{\partial x^i} = N_{(\gamma)i}^{(k)} \frac{\partial \tilde{x}^j}{\partial x^k} \frac{\partial t^\gamma}{\partial \tilde{t}^\beta} - \frac{\partial \tilde{x}_\beta^j}{\partial x^i}. \end{cases}$$

iv) *To give a nonlinear connection \mathcal{H} on E is equivalent to give a set of local functions $\Gamma = (M_{(\beta)\alpha}^{(j)}, N_{(\beta)i}^{(j)})$ which transform by 3.7.*

Example 3.1 Studying the transformation rules of the local components

$$(3.8) \quad \begin{cases} M_{(\beta)\alpha}^{(j)} = -H_{\alpha\beta}^\gamma x_\gamma^j \\ N_{(\beta)i}^{(j)} = \gamma_{ik}^j x_\beta^k, \end{cases}$$

we conclude that $\Gamma_0 = (M_{(\beta)\alpha}^{(j)}, N_{(\beta)i}^{(j)})$ represents a nonlinear connection on E , which is called the *canonical nonlinear connection attached to the semi-Riemannian metrics $h_{\alpha\beta}$ and φ_{ij}* .

Let us consider the 1-form $\delta x_\alpha^i = dx_\alpha^i + M_{(\alpha)\beta}^{(i)} dt^\beta + N_{(\alpha)j}^{(i)} dx^j$. One easily deduces that the set of 1-forms $\{dt^\alpha, dx^i, \delta x_\alpha^i\}$ is a basis in the set of 1-forms.

Definition 3.2 The basis $\left\{ \frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x_\alpha^i} \right\} \subset \mathcal{X}(E)$ and its dual basis $\{dt^\alpha, dx^i, \delta x_\alpha^i\} \subset \mathcal{X}^*(E)$ are called the *adapted bases* on E , determined by the nonlinear connection Γ .

The big advantage of the adapted bases is that the transformation laws of its elements are simple and natural.

Proposition 3.2 *The transformation laws of the elements of the adapted bases attached to the nonlinear connection Γ are*

$$(3.9) \quad \begin{cases} \frac{\delta}{\delta t^\alpha} = \frac{\partial \tilde{t}^\beta}{\partial t^\alpha} \frac{\delta}{\delta \tilde{t}^\beta} \\ \frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{x}^j} \\ \frac{\partial}{\partial x_\alpha^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial t^\alpha}{\partial \tilde{t}^\beta} \frac{\delta}{\delta \tilde{x}_\beta^j}, \end{cases}$$

$$(3.10) \quad \begin{cases} dt^\alpha = \frac{\partial t^\alpha}{\partial \tilde{t}^\beta} d\tilde{t}^\beta \\ dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j \\ \delta x_\alpha^i = \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{t}^\beta}{\partial t^\alpha} \delta \tilde{x}_\beta^j. \end{cases}$$

Remark 3.2 The simple transformation rules 3.9 and 3.10 determine us to describe the objects with geometrical and physical meaning from the subsequent (generalized) metrical multi-time Lagrange theory of physical fields [8], [10], in adapted components. In a such prospect, we emphasize that, using adapted bases of a nonlinear connection Γ , a d-tensor $D = (D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\mu)\dots})$ on E can be regarded as a global geometrical object, locally defined by

$$(3.11) \quad D = D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\mu)\dots} \frac{\delta}{\delta t^\alpha} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial x_\beta^j} \otimes dt^\gamma \otimes dx^k \otimes \delta x_\mu^l \otimes \dots$$

The utilization of certain indices between parenthesis in the description of the local components of the d-tensor D is suitable for contractions. To illustrate this fact, we consider, for example, the local components of the metrical d-tensor 2.2 from the example 2.1. These define the geometrical object

$$(3.12) \quad G = G_{(i)(j)}^{(\alpha)(\beta)} \delta x_\alpha^i \otimes \delta x_\beta^j.$$

On the other hand, considering the local components of the h -normalization d-tensor $J_{(\alpha)\beta j}^{(i)}$, we obtain the representative object

$$(3.13) \quad J = J_{(\alpha)\beta j}^{(i)} \frac{\delta}{\delta x_\alpha^i} \otimes dt^\beta \otimes dx^j.$$

Finally, let us study the relation between the notion of spray and the nonlinear connection. In this context, the coefficients $M_{(\beta)\alpha}^{(j)}$ (resp. $N_{(\beta)i}^{(j)}$) of the nonlinear connection Γ are called the *temporal* (resp. *spatial*) *nonlinear connection*. In this terminology, using the transformation formulas 2.4, 2.6 and 3.7, we can easily prove the following statements.

Theorem 3.3 *i) If $M_{(\alpha)\beta}^{(i)}$ are the components of a temporal nonlinear connection, then the components*

$$(3.14) \quad H_{(\alpha)\beta}^{(i)} = \frac{1}{2} M_{(\alpha)\beta}^{(i)}$$

represent a temporal spray.

ii) Conversely, if $H_{(\alpha)\beta}^{(i)}$ are the components of a temporal spray, then

$$(3.15) \quad M_{(\alpha)\beta}^{(i)} = 2H_{(\alpha)\beta}^{(i)}$$

are the components of a temporal nonlinear connection.

Theorem 3.4 *i) If $G_{(\alpha)\beta}^{(i)}$ are the components of a spatial spray and $G^i = h^{\alpha\beta} G_{(\alpha)\beta}^{(i)}$ represent the h -trace of this spray, then the coefficients*

$$(3.16) \quad N_{(\alpha)j}^{(i)} = \frac{\partial G^i}{\partial x_\gamma^j} h_{\gamma\alpha}$$

represent a spatial nonlinear connection.

ii) Conversely, the spatial nonlinear connection $N_{(\alpha)j}^{(i)}$ induces the spatial spray

$$(3.17) \quad 2G_{(\alpha)\beta}^{(i)} = N_{(\alpha)j}^{(i)} x_\beta^j.$$

Remark 3.3 The previous theorems allow us to conclude that a multi-time dependent spray (H, G) induces naturally a nonlinear connection Γ on E , which is called the *canonical nonlinear connection associated to the multi-time dependent spray (H, G)* . We point out that the canonical nonlinear connection Γ attached to the multi-time dependent spray (H, G) is a natural generalization of the canonical nonlinear connection N induced by a time-dependent spray G from the classical rheonomic Lagrangian geometry [6].

4 Jet prolongation of vector fields

A general vector field X^* on $J^1(T, M)$ can be written under the form

$$X^* = X^\alpha \frac{\partial}{\partial t^\alpha} + X^i \frac{\partial}{\partial x^i} + X_{(\alpha)}^{(i)} \frac{\partial}{\partial x_\alpha^i},$$

where the components X^α , $X^i X_{(\alpha)}^{(i)}$ are functions of $(t^\alpha, x^i, x_\alpha^i)$.

The prolongation of a vector field X on $T \times M$ to a vector field on the 1-jet bundle $J^1(T, M)$ was solved by Olver [12] in the following sense.

Definition 4.1 Let X be a vector field on $T \times M$ with corresponding (local) one-parameter group $\exp(\varepsilon X)$. The *1-th prolongation* of X , denoted by $pr^{(1)}X$, will be a vector field on the 1-jet space $J^1(T, M)$, and is defined to be the infinitesimal generator of the corresponding prolonged one-parameter group $pr^{(1)}[\exp(\varepsilon X)]$, i. e. ,

$$(4.1) \quad [pr^{(1)}X](t^\alpha, x^i, x_\alpha^i) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} pr^{(1)}[\exp(\varepsilon X)](t^\alpha, x^i, x_\alpha^i).$$

In order to write the components of the prolongation, Olver used the α -th *total derivative* D_α of an arbitrary function $f(t^\alpha, x^i)$ on $T \times M$, which is defined by the relation

$$(4.2) \quad D_\alpha f = \frac{\partial f}{\partial t^\alpha} + \frac{\partial f}{\partial x^i} x_\alpha^i.$$

Thus, starting with $X = X^\alpha(t, x) \frac{\partial}{\partial t^\alpha} + X^i(t, x) \frac{\partial}{\partial x^i}$ like a vector field on $T \times M$, Olver introduced the 1-th prolongation of X as the vector field

$$(4.3) \quad pr^{(1)}X = X + X_{(\alpha)}^{(i)}(t^\beta, x^j, x_\beta^j) \frac{\partial}{\partial x_\alpha^i},$$

where

$$X_{(\alpha)}^{(i)} = D_\alpha X^i - (D_\alpha X^\beta) x_\beta^i = \frac{\partial X^i}{\partial t^\alpha} + \frac{\partial X^i}{\partial x^j} x_\alpha^j - \left(\frac{\partial X^\beta}{\partial t^\alpha} + \frac{\partial X^\beta}{\partial x^j} x_\alpha^j \right) x_\beta^i.$$

If we assume that is given a nonlinear connection $\Gamma = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)})$ on $J^1(T, M)$, then the α -th total derivative used by Olver can be written as

$$(4.4) \quad D_\alpha f = \frac{\delta f}{\delta t^\alpha} + \frac{\delta f}{\delta x^i} x_\alpha^i,$$

and, consequently, $D_\alpha f$ represent the local components of a distinguished 1-form on $J^1(T, M)$, which is expressed by $Df = (D_\alpha f) dt^\alpha$.

Now, let there be given a vector field X on $T \times M$. From a geometrical point of view, we can define a 1-jet prolongation of X as the *horizontal lift* X^H of X . This is defined by

$$(4.5) \quad X^H = X^\alpha \frac{\delta}{\delta t^\alpha} + X^i \frac{\delta}{\delta x^i} = X - (M_{(\beta)\alpha}^{(j)} X^\alpha + N_{(\beta)i}^{(j)} X^i) \frac{\partial}{\partial x_\beta^j}.$$

Open problem.

Study the prolongations of vectors, 1-forms, tensors, G -structures from $T \times M$ to $J^1(T, M)$.

Acknowledgments. A version of this paper was presented at Third Conference of Balkan Society of Geometers, Politehnica University of Bucharest, Romania, July 31-August 3, 2000. It is a pleasure for us to thank to Prof. Dr. D. Opreş and to the reviewers of Journal of the London Mathematical Society for their valuable comments upon the previous version of this paper.

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